Dynamic Transformation Method for Modal Synthesis

EDWARD J. KUHAR* AND CLYDE V. STAHLE†
General Electric Company, Valley Forge, Pa.

This paper presents a condensation method for large discrete parameter vibration analysis of complex structures that greatly reduces truncation errors and provides accurate definition of modes in a selected frequency range. A dynamic transformation is obtained from the partitioned equations of motion that relates modes not explicitly in the condensed solution to the retained modes at a selected system frequency. The generalized mass and stiffness matrices, obtained with existing modal synthesis methods, are reduced using this transformation and solved. Revised solutions are then obtained using new transformations at the calculated eigenvalues and are also used to assess the accuracy of the results. If all the modes of interest have not been obtained, the results are used to select a new set of retained coordinates and a new transformation frequency and the procedure repeated for another group of modes. Computations are made tractable by simplified forms of the transformation that result with various modal synthesis methods. Three examples using the dynamic transformation in conjunction with a General Electric stiffness coupling method and the method of Craig and Bampton indicate large reductions in truncation errors and demonstrate the method for sequential groups of modes. Comparisons with truncated results using current methods indicate that two to three times as many accurate modes are obtained from solutions keeping less than half the component modes.

Nomenclature

$[k]$ = stiffness matrix for total structure in $\{x\}$ physical coordinates
$[K]$ = generalized stiffness matrix for total structure in $\{q\}$ modal coordinates
$[k_{CPL}]$ = stiffness matrix in $\{x\}$ coordinates for coupling substructures
$[m]$ = mass matrix for total structure in $\{x\}$ physical coordinates
$[M]$ = generalized mass matrix for total structure in $\{q\}$ modal coordinates
$[\Delta m]$ = incremental mass matrix of coupling structures in $\{x\}$ coordinates
n = number of degrees of freedom
N = number of degrees of freedom in total structure
p = reduction circular frequency
$\{q\}$ = generalized modal coordinates
$[R]$ = transformation matrix defining the relationship of the reduced coordinates, $\{q^R\}$, to be kept coordinates $\{q^K\}$
$[T]$ = dynamic transformation matrix
$\{x\}$ = physical coordinates
$[\alpha]$ = transformation matrix containing substructure constraint and normal modes in $\{x\}$ coordinates relating $\{x\}$ and $\{\eta\}$
$[\beta]$ = transformation matrix relating $\{\eta\}$ and $\{q\}$
$[\gamma]$ = matrix of eigenvectors for total structure in $\{q\}$ modal coordinates
$\{\eta\}$ = substructure modal coordinates defined by $[\alpha]$
λ = system eigenvalue
$[\Phi]$ = matrix of eigenvectors for total structure in $\{x\}$ physical coordinates
ω = substructure circular frequency
Ω = system circular frequency

Subscripts

1 = substructure 1
2 = substructure 2
i = refers to the ith subset or term

Superscripts

B = boundary coordinates
C = constraint modes relating boundary and interior coordinates
I = interior coordinates not on a boundary
K = kept coordinates
N = normal modes of constrained substructure for interior coordinates
R = reduced coordinates
$\bar{}$ = revised value
$\ddot{}$ = second time derivative

Presented as Paper 73-396 at the AIAA/ASME/SAE 14th Structures, Structural Dynamics, and Materials Conference, Williamsburg, Va., March 20–22, 1973; submitted March 29, 1973; revision received October 17, 1973. This research was performed for the NASA-Marshall Space Flight Center.

Index category: Structural Dynamic Analysis.

* Engineer—Spacecraft Dynamics. Member AIAA.
† Manager—Spacecraft Dynamics. Member AIAA.

Introduction

VIBRATION analysis of complex structures is readily accomplished using discrete parameter models. However, when large systems having a broad spectrum of modal frequencies are investigated or relatively high frequency excitations are of concern, the degrees of freedom (DOF) required to accurately represent the structure can be on the order of 500 to 1000 and far exceed the core capacity of the computer. Although computer programs such as NASTRAN have been created specifically to manipulate and solve large complex models, the eigenvalue solution still requires considerable computer run time to obtain the desired results.[1] Current approaches to solving the problem include iterative techniques which use the complete analytical model, condensation methods which reduce the order of the eigenvalue problem for computational efficiency, and modal synthesis. Modal synthesis or a combination of modal synthesis and a condensation method appears particularly attractive for reducing computer requirements.

Modal synthesis methods have been developed to determine the low frequency modes of complex structures.[2] These methods synthesize the vibration modes of the complete structure from substructure analyses. By selecting constraint modes and/or the low frequency natural vibration modes of the substructures (component modes), the order of the eigenvalue solution is reduced. All of these methods are approximate in that the effects of the truncated high frequency component modes are not included in the solution. The accuracy of the solution is assessed and improved by including more modes in the analysis.[3]

Existing modal synthesis methods vary in formulation and in the type of component modes used to represent the substructures. In the methods of Hurty, Craig and Bampton, and Bajan and Feng, substructure natural modes constrained at the attachment points to other substructures are used in conjunction with rigid body and/or constraint modes.[4–6] These "fixed-constraint" modes are well conditioned so that truncation errors resulting

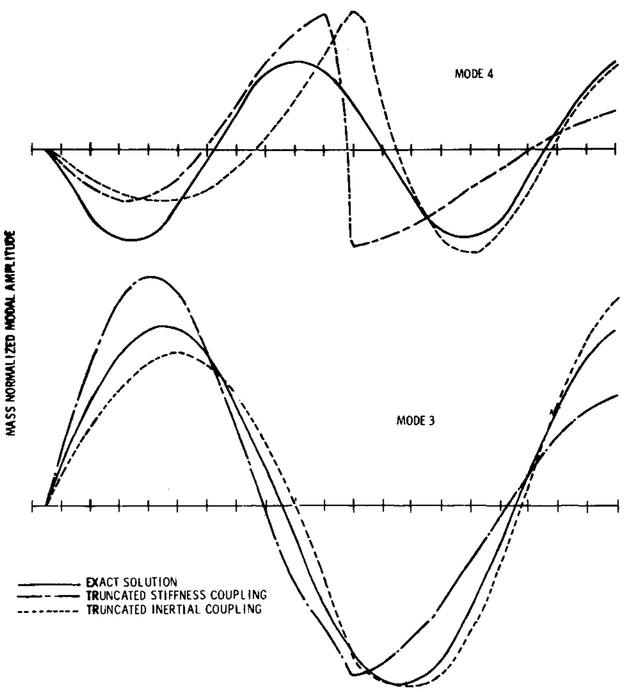

Fig. 1 Effect of truncation on higher modes of longitudinal rod.

from omission of the higher modes are minimized but considerable complexity is introduced by the constraint modes. Because the modal coupling occurs through the mass matrix with these methods, this type of modal synthesis will be referred to as "inertial coupling." In the methods of Goldman and Hou, the use of free-free component modes eliminates the need for constraint modes but results in solutions which are more susceptible to truncation error.[7,8] All of the aforementioned methods assemble the structure by imposing compatibility relations on the attachment coordinates contained in each substructure. When a large number of attachment points are required, the order of the eigenvalue problem is inherently large because the number of constraint modes or free-free component modes must equal the number of attachment coordinates.

A stiffness coupling method that eliminates the need to retain a large number of coordinates because of the attachments has been used extensively at the General Electric Space Division.[9-11] This method uses free-free component vibration modes which are interconnected by flexible links so that there are no common attachment points in the substructures. The assembly of the generalized stiffness parallels that of the displacement method of structural analysis. However, it is susceptible to truncation errors in much the same manner as the methods of Goldman and Hou.

Several other modal synthesis methods have been developed with the intent of reducing truncation errors. The method of Bajan, Feng and Jaszlics, uses successive solutions with different component modes to improve the accuracy of a set of object modes.[12] Its iterative nature and the difficulty in selecting component modes for successive iterations make it somewhat unattractive. In the method of Benfield and Hruda, the use of branch modes with interface loading was shown to improve low frequency modes.[13] Although this method eliminates the need to retain a large number of DOF's for attachment purposes, additional eigenvalue solutions are required prior to synthesizing the structure and the effects on the higher solution modes are unpredictable. Finally, the Guyan reduction method, a "static" transformation method, can be used to improve the low frequency modes by reducing the higher frequency modes from the generalized mass and stiffness matrices but results in increased errors in the higher solution modes.[14] An improved method which reduces truncation errors is needed.

The approach taken in this paper is to use a dynamic transformation method which includes the effects of modes not retained explicitly in the eigenvalue solution. A dynamic transformation that relates the unused coordinates to the retained coordinates at a selected system frequency is obtained from the complete equations of motion. This transformation is then used to reduce the mass and stiffness matrices while retaining those coordinates of primary interest. After solving the reduced eigenvalue problem, the solutions are revised using new transformations at the calculated eigenvalues. Comparison of the revised and initial reduced solutions are used to assess the accuracy of the modes. If all of the modes of interest have not been obtained, the results are used to select a new set of retained coordinates and reduction transformation; and then, the procedure is repeated. The transformation can be used with any of the basic modal synthesis methods. Because the transformation can be at any system frequency, the method is not restricted to low mode solutions but can be used for modes in any frequency range. If the system frequency is selected to be zero, the transformation becomes that of Guyan.

An example of the truncation errors with which we are concerned is shown in Fig. 1. The third and fourth mode shapes from the exact solution of a 20 DOF longitudinal rod are compared with truncated modal synthesis results. Large truncation errors are evident in both mode shapes. It will be shown that the dynamic transformation will reduce these errors to the point that they are not discernible on the graph.

Dynamic Transformation Method

For large models with many DOF, direct solution of the complete systems of equations of motion may become impractical. Current modal synthesis methods provide an approach to solving the large eigenvalue problem by dividing a large structure into several substructures. Using results from the vibration analysis of the individual substructures and imposing equations of equilibrium and compatibility between substructure interfaces enables the substructures to be transformed and coupled modally by a generalized mass matrix, $[M]$, and generalized stiffness matrix, $[K]$. The coordinates, $\{q\}$, corresponding to some transformed set of physical or modal substructure coordinates are partitioned into two groups, kept and truncated. The truncated coordinates are the higher frequency substructure modes and are completely omitted from the equation of motion. As a result, the solutions will have "truncation" errors introduced.

Instead of truncating or omitting modes, all modes will be included through a transformation that relates the "reduced" modes not contained explicitly in the solution to the modes that are "kept." If Ω_i^2 corresponds to an exact eigenvalue of Eq. (1), the relationship between the eigenvalue and its eigenvector may be expressed in terms of the partitioned kept, $\{q^K\}$, coordinates as

$$\Omega_i^2 \begin{bmatrix} M^{KK} & M^{KR} \\ M^{RK} & M^{RR} \end{bmatrix} \begin{Bmatrix} q^K \\ q^R \end{Bmatrix} = \begin{bmatrix} K^{KK} & K^{KR} \\ K^{RK} & K^{RR} \end{bmatrix} \begin{Bmatrix} q^K \\ q^R \end{Bmatrix} \tag{1}$$

Solving this equation for the $\{q^R\}$ reduced coordinates in terms of the $\{q^K\}$ kept coordinates for some general frequency, p, yields the relationship

$$\begin{aligned} \{q^R\} &= -[K^{RR} - p^2 M^{RR}]^{-1}[K^{RK} - p^2 M^{RK}]\{q^K\} \\ &= [R]\{q^K\} \end{aligned} \tag{2}$$

The dynamic transformation matrix, $[T]$, for some "reduction frequency," p, is then defined as

$$[T] = \begin{bmatrix} I \\ R \end{bmatrix} \tag{3}$$

where the expression for $[R]$ is defined by Eq. (2) and I is the identity matrix. For the special case where $p = 0$, the dynamic transformation reduces to the familiar static transformation matrix described by Guyan.

The reduced equation of motion is obtained directly by substituting the coordinate transformation

$$\{q\} = \begin{Bmatrix} q^K \\ \cdots \\ q^R \end{Bmatrix} = [T]\{q^K\} \tag{4}$$

into Eq. (1) and premultiplying by the transpose of $[T]$. Conventional methods of determining eigenvalues may be applied to the reduced equation of motion to obtain a set of frequencies, Ω^K, and a corresponding set of mass normalized eigenvectors, γ^K. From the coordinate relationship defined in Eq. (2), the reduced eigenvectors, $[\gamma^R]$, corresponding to the $\{q^R\}$ reduced coordinates are given by

$$[\gamma^R] = [R][\gamma^K] \tag{5}$$

and the eigenvectors of Eq. (1) may be written in partitioned form as

$$[\gamma^{KR}] = \begin{bmatrix} \gamma^K \\ \cdots \\ \gamma^R \end{bmatrix} \tag{6}$$

This solution will be exact for any Ω_i^K which is the same as the reduction frequency, p, used in developing $[T]$.

Significant improvement can be obtained in the modes and frequencies by applying the Rayleigh-Ritz method in conjunction with the dynamic transformation for individual modes. This part of the dynamic transformation method will be referred to as backsubstitution. For each Ω_i^K to be considered, a revised mode shape, $[\bar{\gamma}^{KR}]_i$, can be determined by substituting $p = \Omega_i^K$ in $[R]$ and mass normalizing the mode shape. A new estimate of Ω_i^K will be then given by

$$\bar{\Omega}_i^{K^2} = [\bar{\gamma}^{KR}]_i^T [K][\bar{\gamma}^{KR}]_i \tag{7}$$

Indications as to the accuracy of the solution are provided by the eigenvalue ratio, $(\Omega_i^{K^2}/\bar{\Omega}_i^{K^2})$, and the normalization factor, i.e., the scalar quantity used to mass normalize the eigenvector. For an exact solution, both factors are unity.

Based upon the component mode coefficients (participation factors) defined by Eq. (6), a new set of kept coordinates and a corresponding reduction frequency, p, can be selected which better approximate the desired solution.

Dynamic Transformation with Stiffness Coupling

The stiffness coupling method of modal synthesis assembles the complete structure in the same manner as the displacement method for structural analysis. The structure is represented by a number of substructures connected through flexible links as indicated by the models shown in Fig. 2. Each substructure is analyzed without the flexible links to determine the component vibration modes with free attachment coördinates. The flexible links are represented by a stiffness matrix relating the interface forces from one set of substructure attachment coordinates to another. As a result of stiffness coupling, substructures can be assembled independently because the coordinates belonging to one particular substructure are not common to any other component substructure. For simplicity, the development of modal synthesis by stiffness coupling will be treated by considering only two substructures.

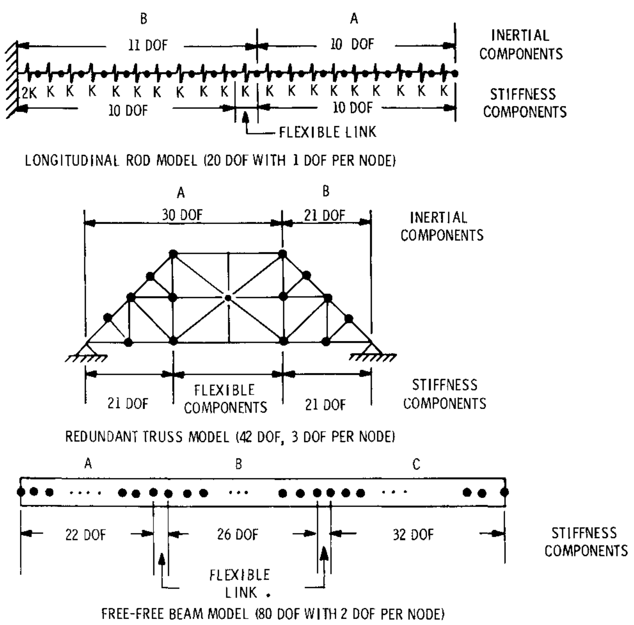

Fig. 2 Models and substructures used for analysis.

For each substructure, a set of eigenvalues, $\lceil \omega_i^2 \rfloor$, and a set of mass normalized eigenvectors, $[\phi_i]$, are determined. The uncoupled equations of motion of two substructures can be represented by one partitioned equation in terms of the $\{x_1\}$ and $\{x_2\}$ physical coordinates of the substructures as

$$\begin{bmatrix} m_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \cdots \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ \cdots \\ x_2 \end{Bmatrix} = 0 \tag{8}$$

Let $[k_{\text{CPL}}]$ represent the stiffness matrix used to relate the interface forces between the connecting DOF of each substructure, and $[\Delta m]$ the incremental mass and inertia of the flexible link. The coupled equations of motion can then be expressed as

$$\left\{ \begin{bmatrix} m_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & m_2 \end{bmatrix} + [\Delta m] \right\} \begin{Bmatrix} \ddot{x}_1 \\ \cdots \\ \ddot{x}_2 \end{Bmatrix} + \left\{ \begin{bmatrix} k_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & k_2 \end{bmatrix} + [k_{\text{CPL}}] \right\} \begin{Bmatrix} x_1 \\ \cdots \\ x_2 \end{Bmatrix} = 0 \tag{9}$$

Solution of Eq. (9) is easily accomplished by a transformation to modal coordinates, $\{q\}$, defined by

$$\{x\} = \begin{Bmatrix} x_1 \\ \cdots \\ x_2 \end{Bmatrix} = \begin{bmatrix} \phi_1 & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \phi_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ \cdots \\ q_2 \end{Bmatrix} = [\phi]\{q\} \tag{10}$$

Substituting the transformation for $\{x\}$ into Eq. (9) and premultiplying by $[\phi]^T$ yields the familiar equation of motion, Eq. (4), in modal coordinates; and the system eigenvectors will be given by

$$[\Phi] = [\phi][\gamma] \tag{11}$$

If care is taken to include the mass effects of the flexible link, $[k_{\text{CPL}}]$, in each of the substructures, only one simplified matrix triple-product is necessary to obtain the generalized mass and stiffness matrices. Noting the orthogonality relationships for the substructure normal modes, the generalized mass and stiffness matrices may simply be expressed as

$$[M] = [I]$$

$$[K] = \begin{bmatrix} \omega_1^2 & \vdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \vdots & \omega_2^2 \end{bmatrix} + [\phi]^T [k_{\text{CPL}}][\phi] \tag{12}$$

where the rows of $[\phi]$ contain only those coordinates to be coupled by $[k_{\text{CPL}}]$. The resulting matrices only need to be assembled in terms of the $\{q^K\}$ and $\{q^R\}$ coordinates before applying the dynamic transformation. Partitioning the eigenvectors into sets of kept and reduced modes before determining the $[K]$ matrix, produces matrices of reduced size easily manipulated by the computer. Defining the substructure vectors as

$$[\phi_1] = [\phi_1^K \vdots \phi_1^R] \qquad [\phi_2] = [\phi_2^K \vdots \phi_2^R]$$

$$[\phi^K] = \begin{bmatrix} \phi_1^K & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \phi_2^K \end{bmatrix} \qquad [\phi^R] = \begin{bmatrix} \phi_1^R & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & \phi_2^R \end{bmatrix} \tag{13}$$

the final form of $[K]$ will automatically be formed in terms of the kept and reduced coordinates and the $[k_{\text{CPL}}]$ terms from Eq. (12) will be given by

$$[\phi]^T [k_{\text{CPL}}][\phi] = \begin{bmatrix} \phi^{K^T} k_{\text{CPL}} \phi^K & \vdots & \phi^{K^T} k_{\text{CPL}} \phi^R \\ \cdots & & \cdots \\ \phi^{R^T} k_{\text{CPL}} \phi^K & \vdots & \phi^{R^T} k_{\text{CPL}} \phi^R \end{bmatrix} \tag{14}$$

Due to the form of the equation of motion for stiffness coupling, $[T]$ reduces to a simplified form where

$$[R] = -[K^{RR} - p^2 I]^{-1} [K^{RK}] \tag{15}$$

and the reduced generalized mass and stiffness matrices can be written in partitioned form as

$$\begin{aligned} [M^K] &= [I] + [R]^T [R] \\ [K^K] &= [K^{KK}] + 2[R]^T [K^{RK}] + [R]^T [K^{RR}][R] \end{aligned} \tag{16}$$

In addition to using normalization factors and the frequency change as a measure of accuracy, the participation factors, $[\gamma^{KR}]$, defined by Eq. (6) are used to determine whether or not a good set of kept modes were selected for the desired solution. Large component mode coefficients in the $[\gamma^R]$ set may indicate a poor choice of coordinates and a deficient result. For the Ω_i^K frequencies accepted as solutions, the partial set of corresponding eigenvectors are obtained by substituting $[\gamma^{KR}]$ for $[\gamma]$ into Eq. (11).

Dynamic Transformation with Inertial Coupling

The inertial coupling method of modal synthesis selected for solution with the dynamic transformation is that of Craig and Bampton.[5] This method is one of the types of inertial coupling methods where the substructures are attached through common points.[4,6] The substructure mass and stiffness matrices, $[m]$ and $[k]$, are expressed in terms of "boundary" and "internal" coordinates. The constraint modes, $[\phi^C]$, and component normal modes, $[\phi^N]$, define a coordinate transformation for each substructure from physical coordinates to modal coordinates defined by

$$\{x_i\} = [\alpha_i]\{\eta_i\} = \begin{bmatrix} I & 0 \\ \phi^C & \phi^N \end{bmatrix}\{\eta_i\} \tag{17}$$

where the constraint modes are the deflected shapes resulting from separately deflecting the boundary coordinates and the component normal vibration modes are determined with all attachment coordinates fixed. Compatibility conditions are imposed by a coordinate transformation $[\beta]$ from the uncoupled modal coordinate, $\{\eta\}$, to the coupled system coordinates, $\{q\}$, given by

$$\{\eta\} = [\beta]\{q\} \tag{18}$$

Applying the two coordinate transformations defined by Eqs. (17) and (18), to the uncoupled equation of motion, Eq. (8), results in the final coupled equation, where the mass and stiffness matrices are of the form

$$[M] = \begin{bmatrix} M^{BB} & M_1^{BI} & M_2^{BI} \\ M_1^{IB} & I & 0 \\ M_2^{IB} & 0 & I \end{bmatrix}$$
$$[K] = \begin{bmatrix} K^{BB} & 0 & 0 \\ 0 & \diagdown\omega_1^2\diagdown & 0 \\ 0 & 0 & \diagdown\omega_2^2\diagdown \end{bmatrix} \tag{19}$$

where the "B" superscripts refer to boundary coordinates and the "I" superscripts to the internal coordinates. The system eigenvectors will be given by

$$[\Phi] = [\alpha][\beta][\gamma] \tag{20}$$

Because of the form of the $[M]$ and $[K]$, $[T]$ reduces to a simplified form different than that for stiffness coupling where

$$[R] = p^2[K^{RR} - p^2 I]^{-1}[M^{RK}] \tag{21}$$

If the boundary points are all assembled into the $\{q^K\}$ coordinate set, the transformation is extremely simple to calculate since the term $[K^{RR} - p^2 I]$ will be a diagonal matrix. For cases when some or all of the boundary points occur in the $\{q^R\}$ coordinate set, $[K^{RR}]$ can be assembled in a form similar to $[K]$ in Eq. (19) and only the $[K^{BB}]$ partition need be inverted for the computation of $[R]$.

The reduced mass and stiffness matrices can be written as

$$[M^K] = [M^{KK}] + 2[R]^T[M^{RK}] + [R]^T[R]$$
$$[K^K] = [K^{KK}] + [R]^T[K^{RR}][R] \tag{22}$$

where $[K^{RR}]$ may be completely diagonal or contain a subportion of the boundary coordinates. The backsubstituted eigenvectors are easily obtained if all the boundary coordinates are contained in $\{q^K\}$. However, unlike stiffness coupling, only those coordinates of $[\gamma^{KR}]$, corresponding to the internal coordinates may be used as participation factors in selecting modal contributors.

Numerical Results

The dynamic transformation method with stiffness and/or inertial coupling was used to solve three structural models, Fig. 2. A single interface coordinate was used to connect the two rod substructures while nine redundant coordinates were used with the truss. An 80 DOF uniform beam model was analyzed using three substructures of varying sizes. The inertial coupling substructure model boundaries are at the center of the boundary mass with half the mass contained in each model. The stiffness coupling substructure models include the total boundary mass but do not include the flexible links connecting the substructures together. The component modes were determined with fixed attachment coordinates for inertial coupling and with free attachment coordinates for stiffness coupling.

Three parameters were varied in the analyses: 1) the number of kept and/or reduced coordinates; 2) the reduction frequency, p; and 3) the set of coordinates kept for the eigenvalue solution. The number of kept or reduced coordinates were incremented in steps of approximately 20% of the total structural DOF's with major emphasis on 20% and 40% solutions where major benefits are derived. The reduction frequency was selected between system modes to show general trends, i.e., $p^2 = 1/2(\lambda_n + \lambda_{n+1})$. The coordinates selected as the kept set were selected as a sequential group from the constraint modes and/or the frequency ordered component modes depending on the frequency range of interest.

The results are evaluated using frequency and mode shape errors, and the portion of the calculated modes which satisfy a frequency error criterion. The percentage error in the modal frequency was used to enable comparisons with other studies.[12-14] The error of unit length eigenvectors was evaluated using the standard error of estimate relative to the complete solution

$$[\sum (x_{i_{\text{exact}}} - x_i)^2/N]^{1/2}$$

The portion of the calculated elastic modes having a modal frequency error less than 0.1% was used to summarize and compare results.

Low Mode Results

The effect of the number of modes used in dynamic reduction on the truncation error is shown in Figs. 3 and 4 for the rod. The curves show the results of 4 DOF solutions ($n^K = 0.2N$)

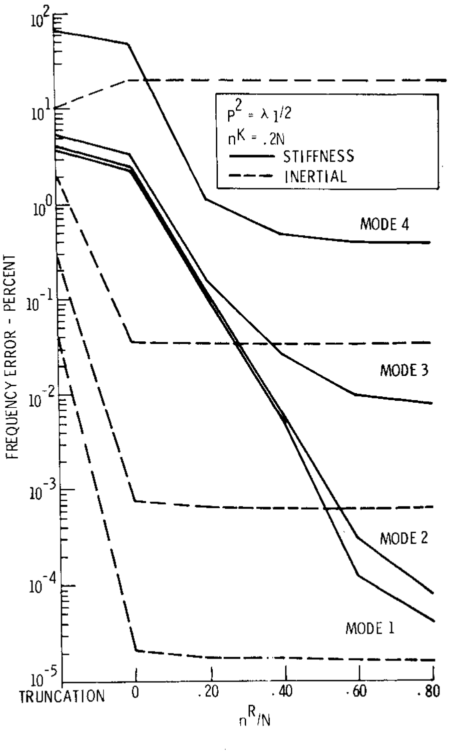

Fig. 3 Frequency error.

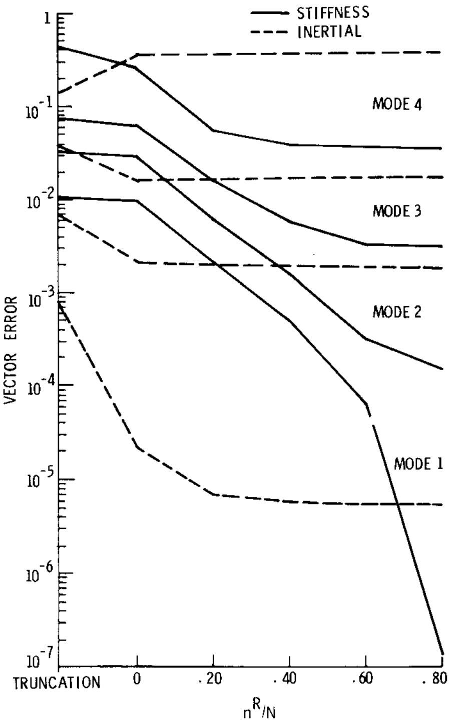

Fig. 4 Vector error.

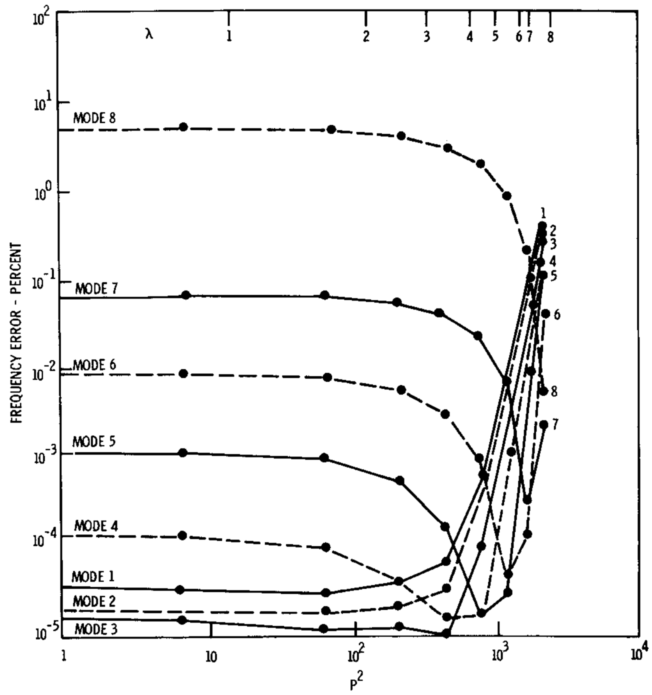

Fig. 5 Frequency error envelope.

with the higher frequency component modes completely truncated. A reduction frequency of $(\lambda_1/2)^{1/2}$ was used. The curves indicate that inertial coupling is improved primarily by back-substitution as indicated by the difference between truncation and 0% reduced. The stiffness coupling results show continuing improvement in all modes as additional modes are included through the dynamic transformation. The vector error appears to be significantly larger than the frequency error for both modal synthesis methods.

A frequency error criterion of 0.1% for accepting a solution appears reasonable if vector errors are not considered. The mode shapes shown in Fig. 1 correspond to the truncated points shown in Figs. 3 and 4. None of the truncated solutions are acceptable. The truncated Mode 3 solutions have frequency errors greater than 2% and maximum modal deflection errors greater than 16%. For the dynamic transformation Mode 3 solutions of Fig. 3, the solutions appear acceptable with maximum modal deflection errors less than 5% and frequency errors less than 0.03%. These results tend to indicate that a frequency error of 0.1% or less corresponds to a maximum modal deflection error of 5%. Therefore, a frequency error criterion of 0.1% has been selected for subsequent comparisons although larger values have been used in the past.

The results of partial reduction analyses are summarized in Table 1 in terms of the portion of modes satisfying a 0.1% frequency error criterion. On the order of $\frac{1}{2}$ to $\frac{7}{8}$ of the dynamic transformation modes satisfy the error criterion as compared to zero to half using truncation. The largest improvement is shown by the solutions for 20% of the modes. The results agree with the trends shown in Fig. 3.

Typical results indicating the effect of the reduction frequency (p) on the frequency error are shown in Fig. 5. The curves are for an eight mode stiffness coupling solution of the rod using all modes through dynamic reduction. Reduction frequencies were selected at average eigenvalue points so that the curves do not show the sharp error reduction obtained near a modal frequency and are representative of error envelopes. If the intent is to calculate as many modes as possible with a frequency error less than 0.1%, a reduction frequency just below the highest modal frequency would be selected and would make seven solution modes acceptable. Results obtained using the best average eigenvalue for reduction are summarized in the "Low Mode" column of Table 2 and, when compared to Table 1, indicate the improvement in the portion of acceptable calculated modes that can be obtained from selection of the reduction frequency.

The effect of increasing the size of the eigenvalue problem while reducing all other modes through the dynamic transformation was examined for the rod, Fig. 6. The curves compare truncated and dynamic transformation results for stiffness and inertial coupling in terms of the portion of calculated modes

Table 1 Portion of calculated low modes with frequency error less than 0.1%, partial reduction

		Solutions for $n^K \simeq 0.2$ N						Solutions for $n^K \simeq 0.4$ N				
			n^R/N						n^R/N			
Model	Method	Truncation	0.0	0.20	0.40	0.60	0.80	Truncation	0.0	0.20	0.40	0.60
Longitudinal Rod	Stiffness	0	0	0.25	0.75	0.75	0.75	0	0	0.75	0.87	0.87
	Inertial	0.25	0.75	0.75	0.75	0.75	0.75	0.37	0.75	0.75	0.75	0.75
Redundant Truss	Stiffness	0	0.37	0.62	0.75	0.75	0.75	0.06	0.37	0.75	0.75	0.75
	Inertial	-	-	-	-	-	-	0.56	0.56	0.56	0.56	0.56
Beam	Stiffness	0	-	-	-	-	0.79*	0.10	-	-	-	0.83*

*Elastic DOF/calculated DOF-2

Table 2 **Portion of calculated modes with frequency error less than 0.1%, all mode solutions**

Model	Method	Solutions for $n^K \simeq 0.2$ N				Solutions for $n^K \simeq 0.4$ N			
			Dynamic Transformation				Dynamic Transformation		
		Truncation	Low Modes	Middle Modes	High Modes	Truncation	Low Modes	Middle Modes	High Modes
Longitudinal Rod	Stiffness	0	1.00	0.50/0.75	1.00	0	0.87	0.62/0.75	0.75
	Inertial, constraint modes kept	0.25	0.75	0.75/1.00	0.75	0.37	0.75	0.87	0.87
	Inertial, no constraint modes kept	-	-	0.75	1.00	-	-	0.75/0.87	0.87
Redundant Truss	Stiffness	0	0.75	0.37	1.00	0.06	0.87	0.75	1.00
	Inertial, constraint modes kept	-	-	-	-	0.56	0.69	0.38/0.69	0.50
	Inertial, no constraint modes kept	-	-	-	-	-	-	0.25	0.62
Free Free Beam	Stiffness	0	0.79*	0.62	0.75	0.10*	0.83*	-	-

*Elastic DOF

that satisfy the 0.1% frequency error criterion. A large improvement is obtained with either coupling method when less than half the system modes are calculated. Because solutions for only a small portion of the modes are generally of interest for large systems, this trend makes the method particularly appealing.

Dramatic improvement in the third and fourth mode solutions of the rod, Fig. 1, were obtained from 4 DOF solutions using the dynamic transformation in different ways. For stiffness coupling, the third mode solution errors become insignificant using a low reduction frequency and reducing only a portion of the modes, Figs. 3 and 4. For inertial coupling, it is necessary to use a reduction frequency near the third mode to obtain vector errors comparable to stiffness coupling although the error in the maximum modal deflection was only 2% with the low reduction frequency. Similarly, a reduction frequency near mode 4 is required to obtain an accurate mode 4 solution with stiffness coupling, e.g., 0.004% frequency error and 0.005 vector standard error for $p^2 = \frac{1}{2}(\lambda_3 + \lambda_4)$. However, with inertial coupling, accurate fourth mode definition required a change in the kept set of coordinates.

Middle and High Mode Results

Analyses to determine the middle and high frequency modes of the models were performed to determine the feasibility of using the dynamic transformation method in this manner. The reduction frequency was selected between modal frequencies as in the previous analyses. A sequential group of frequency ordered component modes was selected as kept coordinates and used either with or without the boundary coordinates. No attempt was made to select an optimum set of kept coordinates.

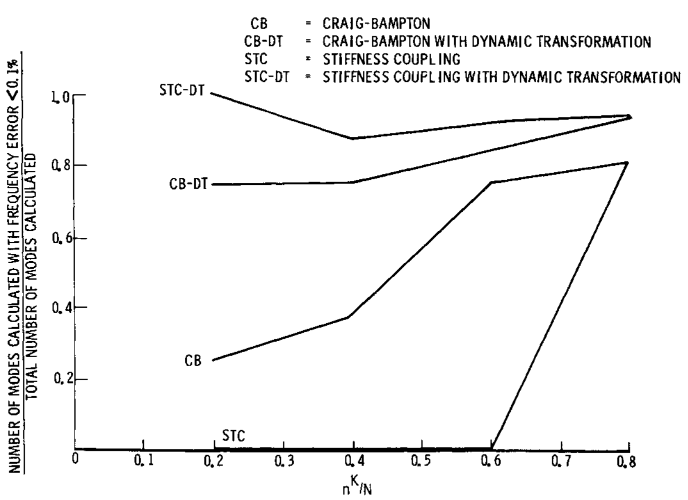

Fig. 6 **Calculated rod modes with frequency error < 0.1%.**

Solutions for 20% and 40% of the modes were calculated reducing all remaining modes through the dynamic transformation.

Table 2 summarizes the results of these analyses and indicates a range of values when more than one middle mode solution was performed. In most analyses, over half the solution modes have frequency errors of less than 0.1%. Inertial coupling can be performed either with or without retaining the boundary coordinates but decreased vector error was observed for solution without the boundary coordinates. Stiffness coupling appears to be slightly better than inertial coupling for intermediate and high mode solutions. These results show the feasibility of using relatively small sequential solutions to obtain the system modes of interest.

Several analyses were performed to determine if intermediate and high mode solutions could be obtained regardless of the kept set of coordinates. The results showed that, if the reduction frequency corresponds to a modal frequency, the vector errors were within computer accuracy regardless of the kept coordinates selected.

Comparisons with Other Methods

The dynamic transformation method is compared to the modal substitution method in Fig. 7.[12] The modal substitution method is an iterative method that introduces sequential groups of subsystem modes into the analysis. The model used in their

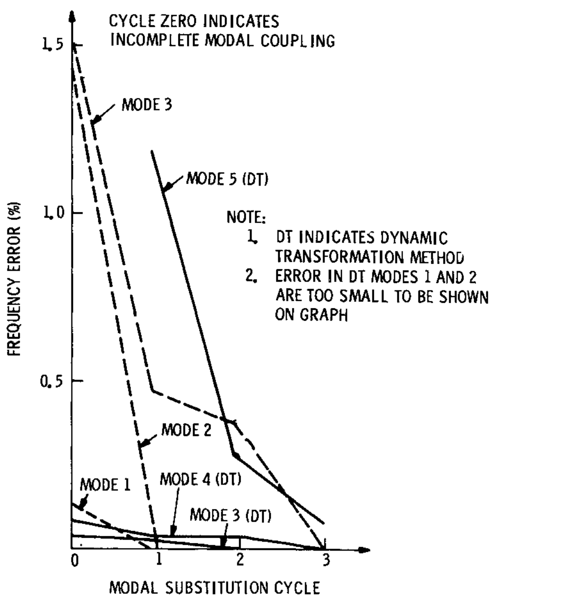

Fig. 7 **Frequency error history comparison.**

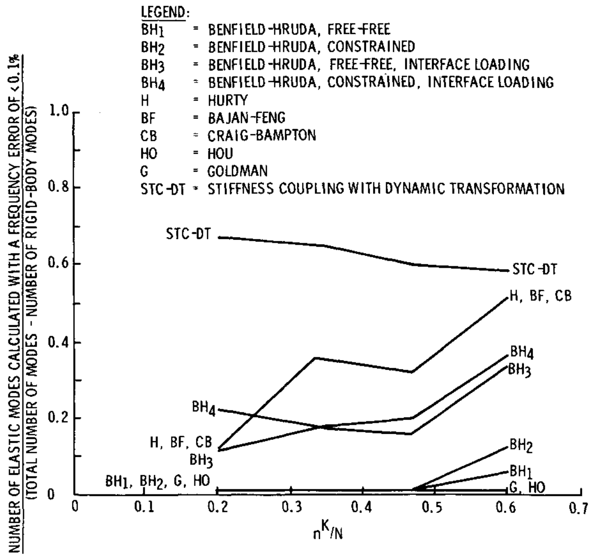

Fig. 8 Comparison of methods with frequency error of 0.1%.

studies were analyzed similarly using stiffness coupling. Five solutions were obtained using p^2 values based on the higher eigenvalues obtained from the first solution. The mode near the p^2 value was obtained within computer accuracy for each solution. The frequency error history of the solutions compares favorably with the modal substitution method.

The results of Benfield, Bodley, and Morosow which compare nine modal synthesis methods were extended to include the dynamic transformation method with stiffness coupling.[15] A 60 DOF planar truss composed of two equal lateral bays and nine equal longitudinal bays was solved using the modes of two substructures as in their paper. A redundant set of six coordinates connect the substructures. For stiffness coupling, the truss was broken into two sections, each having four longitudinal bays (30 DOF). A frequency cut off criterion was used to select the kept coordinates while the other modes were included through the dynamic transformation. The portion of the calculated modes which have a frequency error less than 0.1% is compared with their results in Fig. 8. As in most of the examples presented the number of acceptable modes is doubled or tripled when less than half the modes are kept.

Conclusions

A method of vibration analysis which allows low, intermediate and high mode solutions to be obtained by solving an eigenvalue problem much smaller in size than that of the structural model has been described and numerical results presented. Comparisons with other methods have shown dramatic improvement in the number of modes having small errors. Truncation errors have been shown to be significantly reduced. The method can be applied with any basic modal synthesis technique. It is ideally suited to modal synthesis because of computational simplifications due to special forms of the mass and stiffness matrices and because the conditioning of the problem permits rational selection of coordinates. Several sequential solutions can be used to obtain those modes of interest to the analyst with substantial reductions in computer requirements. For large problems, modes cannot only be reduced through the dynamic transformation, but those modes least affecting the solution can be truncated. Limitations may be imposed on inertial coupling by a large number of attachment coordinates and difficulty in selecting kept coordinates since the coordinates are a combination of constraint and component modes.

As presented, the dynamic transformation method uses backsubstitution to increase the accuracy of all modes. However, farther experience in the application of the method may indicate that it should be used selectively. If some of the modes are known to be accurate (e.g., the lowest modes of a low mode set), further improvement from backsubstitution may not be necessary. On the other hand, if some modes are well removed from the reduction frequency, adequate impovement may not be possible without a change in the reduction frequency and a change in the kept coordinates. There appears to be a middle range where backsubstitution can be used effectively.

The dynamic transformation method has also been applied directly to discrete parameter models. Although some of the models investigated were completely ill-conditioned for coordinate selection (e.g., uniform rod), preliminary results indicate significant improvement over those obtained with Guyan's static reduction. More work is needed to determine the most effective method of applying it to discrete models.

References

[1] Jones, T. C. and Pinson, L. D., "Adaptation of NASTRAN to the Analysis of the Viking Space Vehicle," *NASTRAN User's Experiences*, TMX-2378, Sept. 1971, NASA.

[2] Hart, G. C., Hurty, W. C., and Collins, J. D., "A Survey of Modal Synthesis Methods," *Proceedings of the National Aeronautic and Space Engineering and Manufacturing Meeting*, Los Angeles, Calif., Sept. 28–30, 1971.

[3] Hurty, W. C., "A Criterion for Selecting Realistic Natural Modes of a Structure," TM 33-364, Nov. 1967, Jet Propulsion Lab., Pasadena, Calif.

[4] Hurty, W. C., "Dynamic Analysis of Structural Systems Using Component Modes," *AIAA Journal*, Vol. 3, No. 4, April 1965, pp. 678–685.

[5] Craig, R. R. and Bampton, M. C. C., "Coupling of Substructures for Dynamic Analysis," *AIAA Journal*, Vol. 6, No. 7, July 1968, pp. 1313–1321.

[6] Bajan, R. L. and Feng, C. C., "Free Vibration Analysis by the Modal Substitution Method," AAS Paper 68-8-1, July 1968, Denver, Colo.

[7] Goldman, R. L., "Vibration Analysis by Dynamic Partitioning," *AIAA Journal*, Vol. 7, No. 6, June 1969, pp. 1152–1154.

[8] Hou, S. N., "Review of Modal Synthesis Techniques and a New Approach," *The Shock and Vibration Bulletin*, No. 40, Dec. 1969, Naval Research Lab., Washington, D.C.

[9] Roach, R. E., Jr., "Joining Subsystems Together in Modal Coordinates," PIR 4T45-17, Aug. 1966, General Electric Co., Missile and Spacecraft Dept., Valley Forge, Pa.

[10] Krieger, F. D., "Commode Users Manual, Component Mode Analysis Program," PIR SM8156-71P-3201, Oct. 1966, General Electric Co., Missile and Spacecraft Dept., Valley Forge, Pa.

[11] Cokonis, T. J. and Sardella, G., "Vibration Analysis and Test of the Earth Resources Technology Satellite," *The Shock and Vibration Bulletin*, No. 42 Pt. 2, Jan. 1972, Naval Research Lab., Washington, D.C.

[12] Bajan, R. L., Feng, C. C., and Jaszlics, I. J., "Vibration Analysis of Complex Structural Systems by Modal Substitution," *Proceedings of the 39th Shock and Vibration Symposium*, Monterey, Calif., Oct. 1968.

[13] Benfield, W. A. and Hruda, R. F., "Vibration Analysis of Structures by Component Mode Substitution," *AIAA Journal*, Vol. 9, No. 7, July 1971, pp. 1255–1261.

[14] Guyan, R. J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, Feb. 1965, p. 380.

[15] Benfield, W. A., Bodley, C. S., and Morosow, G., "Modal Synthesis Methods," *Proceedings of Symposium on Substructure Testing and Synthesis*, Aug. 30, 1972, Marshall Space Flight Center, Huntsville, Ala.